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A new method for recovering high-order mixed derivatives of bivariate functions

Yevgeniya Semenova[1], Serhii Solodky[2]

[1] Candidate of Physical and Mathematical Sciences, Institute of Mathematics, National Academy of Sciences of Ukraine, 3, Tereschenkivska Str., 01024, Kiev, e-mail: semenovaevgen@gmail.com
[2] Doctor of Physical and Mathematical Sciences, Institute of Mathematics, National Academy of Sciences of Ukraine, 3, Tereschenkivska Str., 01024, Kiev, e-mail: solodky@imath.kiev.ua

The problem of numerical recovering high-order mixed derivatives of bivariate functions with finite smoothness is studied. On the basis of the truncation method, an algorithm for numerical differentiation is constructed, which guarantees a high order of approximation accuracy.

Ключові слова: *Numerical differentiation, Legendre polynomials, truncation method*

Introduction. Many research activities on the problem of stable numerical differentiation have been taking place due to the importance of this tool in such areas of science and technology as finance, mathematical physics, image processing, analytical chemistry, viscous elastic mechanics, reliability analysis, pattern recognition, and many others. Among this investigation, we highlight [1], which is the first publication on numerical differentiation in terms of the theory of ill-posed problems. Further research [1] has been continued in numerous publications on numerical differentiation for covering different classes of functions and the types of proposed methods. Despite the abundance of works on this topic, the problem of recovery of high-order derivatives was considered only in a few publications. In particular, the results of [2] have opened a perspective for further investigation of numerical methods for the recovery of high-order derivatives. Namely, as the main criteria of the method's efficiency have been taken its ability to achieve the optimal order of accuracy by using a minimal amount of discrete information. Note that particular these aspects of numerical differentiation remain still insufficiently studied. The present paper continues the research of [2], [3] and proposes a numerical method for recovering the high-order mixed derivatives of smooth bivariate functions. The method is not only stable to small perturbations of the input data, but achieves a high order of accuracy with economical use of discrete information, and also has a simple numerical implementation.

1. Description of the problem

Let $\{\varphi_k(t)\}_{k=0}^{\infty}$ be the system of Legendre polynomials orthonormal on $[-1,1]$ as

$\varphi_k(t) = \sqrt{k+1/2}(2^k k!)^{-1} \frac{d^k}{dt^k}[(t^2-1)^k], k = 0,1,2,\ldots$. By $L_2 = L_2(Q)$ we mean space of square-summable on $Q = [-1,1]^2$ functions $f(t,\tau)$ with inner product $\langle f,g\rangle = \int_{-1}^{1}\int_{-1}^{1} f(t,\tau)g(t,\tau)d\tau dt$ and corresponding norm

$$\| f \|_{L_2}^2 = \sum_{k,j=0}^{\infty} |\langle f,\varphi_{k,j}\rangle|^2 < \infty,$$

where $\langle f,\varphi_{k,j}\rangle$ are Fourier-Legendre coefficients of f. Let ℓ_p, $1 \le p \le \infty$, be the space of numerical sequences $\overline{x} = \{x_{k,j}\}_{k,j\in\mathbb{N}_0}$, $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$, such that

$$\| \overline{x} \|_{\ell_p} := (\sum_{k,j\in\mathbb{N}_0} |x_{k,j}|^p)^{\frac{1}{p}} < \infty, 1 \le p < \infty,$$

and $\| \overline{x} \|_{\ell_\infty} = \sup_{k,j\in\mathbb{N}_0} |x_{k,j}| < \infty, p = \infty$. We introduce the class of functions $L_{s,2}^{\mu}(Q) = \{f \in L_2(Q) \colon \| f \|_{s,\mu}^s := \sum_{k,j=0}^{\infty} (\underline{k}\cdot \underline{j})^{s\mu} |\langle f,\varphi_{k,j}\rangle|^s \le 1\}$, where $\mu > 0$, $1 \le s < \infty$, $\underline{k} = max\{1,k\}$, $k = 0,1,2,\ldots$. Note that $L_{s,2}^{\mu}$ is a generalization of a class of bivariate functions with dominating mixed partial derivatives. Moreover, let $C = C(Q)$ be the space of continuous on Q bivariate functions.

We represent a function $f(t,\tau)$ from $L_{s,2}^{\mu}$ as

$$f(t,\tau) = \sum_{k,j=0}^{\infty} \langle f,\varphi_{k,j}\rangle \varphi_k(t)\varphi_j(\tau),$$

and by its mixed derivative we mean the following series

$$f^{(r,r)}(t,\tau) = \sum_{k,j=r}^{\infty} \langle f,\varphi_{k,j}\rangle \varphi_k^{(r)}(t)\varphi_j^{(r)}(\tau), r = 1,2,\ldots. \quad (1)$$

assume that instead of the exact values of the Fourier-Legendre coefficients $\langle f,\varphi_{k,j}\rangle$ only their perturbations are known with the error level δ in the metric of ℓ_p,

$1 \le p \le \infty$.

More accurately, we assume that there is a sequence of numbers

$$\overline{f^\delta} = \{\langle f^\delta,\varphi_{k,j}\rangle\}_{k,j\in\mathbb{N}_0}$$

such that for $\overline{\xi} = \{\xi_{k,j}\}_{k,j\in\mathbb{N}_0}$, where $\xi_{k,j} = \langle f - f^\delta,\varphi_{k,j}\rangle$, and for some $1 \le p \le \infty$ the relation

$$\| \overline{\xi} \|_{\ell_p} \le \delta, 0 < \delta < 1, \quad (2)$$

is true.

The research of this work is devoted to recovering the derivative (1) of functions from $L_{s,2}^{\mu}$. Our goal is to achieve the best order of accuracy by using the minimal number of perturbed Fourier-Legendre coefficients $\langle f^\delta,\varphi_{k,j}\rangle$.

2. Truncation method

It should be noted that at the moment a number of approaches were developed for numerical differentiation. All these methods are accepted to divide into three groups (see [4]): difference methods, interpolation methods and regularization methods. As is known, the first two types of methods have their advantage in the simplicity of implementation, but they guarantee satisfactory accuracy only in the case of exactly given input data about the differentiable function. At the same time, regularization methods give stable approximations to the desired derivatives in the case of perturbed input data but most of them (for example, the Tikhonov method and its various variations) are quite complicated for numerical realization in view of their integral form and require hard-to-implement rules for determination of regularization parameters (see [4]). Recently in [2] a concise numerical method, called the truncation method, has been proposed as a stable and simple approach to numerical differentiation of multivariable functions. The essence of this method is to replace the Fourier series (1) with a finite Fourier sum using perturbed data $\langle f^{\delta}, \varphi_{k,j}\rangle$. In the truncation method, to ensure the stability of the approximation and achieve the required order accuracy, it is necessary to choose properly the discretization parameter, which here serves as a regularization parameter. So, the process of regularization in the method under consideration consists in matching the discretization parameter with the perturbation level δ of the input data. The simplicity of implementation is the main advantage of this method.

In the case of an arbitrary bounded domain Ω of the coordinate plane$[r,\infty)\times[r,\infty)$, the truncation method for differentiating functions of two variables has the form $\mathcal{D}_{\Omega}^{(r,r)} f^{\delta}(t,\tau) = \sum_{(k,j)\in\Omega} \langle f^{\delta}, \varphi_{k,j}\rangle \varphi_k^{(r)}(t)\varphi_j^{(r)}(\tau)$. By $card(\Omega)$ we mean the number of points that make up Ω.

In order to increase the efficiency of the approach under study, we take a hyperbolic cross as the domain Ω of the following form $\Omega = \Gamma_n := \{(k,j): k\cdot j \le rn-1, k,j = r,\dots,n-1\}$,$card(\Gamma_n) = O(nlnn)$. Then our version of the proposed truncation method can be written as

$$\mathcal{D}_{n}^{(r,r)} f^{\delta}(t,\tau) = \sum_{k,j\ge r, kj\le rn-1} \langle f^{\delta}, \varphi_{k,j}\rangle \varphi_k^{(r)}(t)\varphi_j^{(r)}(\tau). \quad (3)$$

3. Error estimate in L_2 – metric

Theorem 1. Let $f \in L_{s,2}^{\mu}$, $1 \le s < \infty$, $\mu > 2r - 1/s + 1/2$, and let the condition (2) be satisfied. Then for $n \asymp \left(\delta^{-1} ln^{1/p-1/s}\frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/s}}$ it holds

$$\| f^{(r,r)} - \mathcal{D}_{n}^{(r,r)} f^{\delta} \|_{L_2} \le c\left(\delta ln^{1/s-1/p}\frac{1}{\delta}\right)^{\frac{\mu-2r+1/s-1/2}{\mu-1/p+1/s}} ln^{3/2-1/s}\frac{1}{\delta}.$$

Corollary 1. In the considered problem, the truncation method $\mathcal{D}_n^{(r,r)}$(3) achieves the accuracy

$$O\left(\left(\delta ln^{1/s-1/p}\frac{1}{\delta}\right)^{\frac{\mu-2r+1/s-1/2}{\mu-1/p+1/s}} ln^{3/2-1/s}\frac{1}{\delta}\right)$$

on the class $L^{\mu}_{s,2}$, $\mu > 2r-1/s+1/2$, and requires

$$\mathrm{card}(\Gamma_n) \asymp n\ln n \asymp (\delta^{-1}\ln^{\mu}\frac{1}{\delta})^{\frac{1}{\mu-1/p+1/s}}$$

perturbed Fourier-Legendre coefficients.

Remark 1 Let's consider the standard variant of the truncation method with $\Omega = \square_n := [r,n]\times[r,n]$. It is easy to verify that such an approach guarantees the accuracy $O\left(\delta^{\frac{\mu-2r+1/s-1/2}{\mu+2r-2/p+1/s+1/2}}\right)$ on the class $L^{\mu}_{s,2}$, $\mu > 2r-1/s+1/2$, and requires $\mathrm{card}(\square_n) \asymp n^2 \asymp \delta^{-\frac{2}{\mu+2r-\frac{2}{p}+\frac{1}{s}+\frac{1}{2}}}$ perturbed Fourier-Legendre coefficients. Comparison of the estimates found above with the corresponding estimates for the method $\mathcal{D}_n^{(r,r)}$ (3) (see Corollary 1) demonstrates that (3) is more efficient both in terms of accuracy and the amount of discrete information used.

3. Error estimate in C – metric

Theorem 2. Let $f \in L^{\mu}_{s,2}$, $1 \le s < \infty$, $\mu > 2r-1/s+3/2$, and let the condition (2) be satisfied. Then for $n \asymp \left(\delta^{-1} ln^{1/p-1/s}\frac{1}{\delta}\right)^{\frac{1}{\mu-1/p+1/s}}$ it holds $\| f^{(r,r)} - \mathcal{D}_n^{(r,r)} f^{\delta} \|_C \le c\left(\delta ln^{1/s-1/p}\frac{1}{\delta}\right)^{\frac{\mu-2r+1/s-3/2}{\mu-1/p+1/s}} ln^{2-1/s}\frac{1}{\delta}$.

Corollary 2. In the considered problem, the truncation method $\mathcal{D}_n^{(r,r)}$ (3) achieves the accuracy $O\left(\left(\delta ln^{1/s-1/p}\frac{1}{\delta}\right)^{\frac{\mu-2r+1/s-3/2}{\mu-1/p+1/s}} ln^{2-1/s}\frac{1}{\delta}\right)$ on the class $L^{\mu}_{s,2}$, $\mu > 2r-1/s+3/2$, and requires $card(\Gamma_n) \asymp nlnn \asymp (\delta^{-1} ln^{\mu}\frac{1}{\delta})^{\frac{1}{\mu-1/p+1/s}}$ perturbed Fourier-Legendre coefficients.

Remark 2 Consider the standard variant of the truncation method with $\Omega = \square_n := [r,n]\times[r,n]$. It is easy to verify that this approach guarantees the accuracy $O\left(\delta^{\frac{\mu-2r+1/s-3/2}{\mu+2r-2/p+1/s+3/2}}\right)$ on the class $L^{\mu}_{s,2}$, $\mu > 2r-1/s+3/2$, and requires $card(\square_n) \asymp n^2 \asymp \delta^{-\frac{2}{\mu+2r-\frac{2}{p}+\frac{1}{s}+\frac{3}{2}}}$ perturbed Fourier-Legendre coefficients. Comparison of the estimates found above with the corresponding estimates for the method $\mathcal{D}_n^{(r,r)}$ (3) (see Corollary 2) demonstrates that (3) is more efficient both in terms of accuracy and the amount of discrete information used.

Remark 3. The method $\mathcal{D}_n^{(r,r)}$ (3) was studied earlier (see [5]) for the problem of numerical differentiation of functions from $L^{\mu}_{s,2}$ in the case of $r = 1$ and $p = s = 2$.

In addition, for the recovery of mixed derivatives $f^{(2,2)}$, this method was considered in [3]. Thus, Theorems 1 and 2 generalize previously known results for the case of arbitrary r, p, s.

Remark 4. It is easy to show that method (3) is optimal in terms of the informational complexity of the numerical differentiation problem.

Conclusions. In the work, a new approach to the numerical recovering mixed derivatives of any order of bivariate functions is investigated. For the proposed method, accuracy estimates are found in integral and uniform metrics, and the amount of discrete information used is calculated.

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Новий метод для відновлення мішаних похідних вищих порядків функцій двох змінних

Євгенія Семенова, Сергій Солодкий

Досліджено задачу відновлення мішаних похідних вищих порядків функцій двох змінних зі скінченною гладкістю. На основі методу зрізки побудовано алгоритм чисельного диференціювання, який досягає високого порядку точності.

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